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BARYON STAR MODELS

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BARYON STAR MODELS

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Abstract 26366

We consider, in the framework of general relativity, large masses consisting of identical baryons (whose mass is taken to be that of the neutron). Four assumptions are made as to their interactions, leading to four equations of state, and for each the masses and radii of a number of configurations are calculated, for various values of the central number density. The relation of the mass to a parameter t_o (= $4\sinh^{-1}\left(\frac{\hbar}{m_B c}(3\pi^2 n_o)^{1/3}\right)$, where n_o is the central number density) for the four equations of state are strikingly similar.

author

I. Introduction

Several discussions have been given (e.g., Oppenheimer and Volkoff 1939; Cameron 1959· Saakyan 1963· Ambartsumyan and Saakyan 1961; Ambartsumyan and Saakyan 1960) of the problem of the structure of stellar masses composed of elementary particles at ultra high densities ($\gtrsim$ nuclear density). Such stars are called neutron stars (and are sometimes called hyperon stars). It seems at least possible that such stars exist in nature, presumably as the relics of supernovae. Recent orbital X-ray experiments indicate that such stars might even exist.

In this connection it seemed worthwhile to consider whether certain simple alternatives for the equation of state of the particles could be justified on theoretical grounds, and if so, whether the results would have any similarity to those for a perfect Fermi gas with no interaction.

The first alternative is (potential energy density) $\sim$ (number density)$^{5/3}$. This originated in a hypothesis of Zel'dovich(1959) of a common ferion core "inside" all baryons, which then leads to a strong repulsion (potential $\propto \frac{1}{r^2}$) when the separation is small and the relative angular momentum is zero. The second was (potential energy density) $\propto$ (number density)^2. This originated in a theory of Zel'dovich(1961) in which baryons interact via a vector meson with a mass.

II. The Equations of Hydrostatic Equilibrium

It has been shown that the relativistic equations of hydrostatic equilibrium, are

$$\frac{dM_r}{dr} = 4\pi \frac{1}{c^2} \epsilon r^2$$

$$\frac{dP}{dr} = -\frac{G(\epsilon/c^2 + P/c^2)(M_r + 4\pi r^3 P/c^2)}{r(r - 2GM_r/c^2)} \tag{1}$$

Here M_r is the (proper) mass inside a sphere of radius r; P the pressure; ϵ the (proper) energy density, including the rest energy; c the velocity of light, G Newton's constant of gravitation; and r the distance from the center of the (isotropic) sphere. The equation of state is determined from the microscopic properties of the medium. In the non-relativistic limit $\epsilon \rightarrow \rho c^2$ and $P \ll \rho c^2$, $r \gg \frac{GM}{c^2}$, then Eqns.(1) reduce to the familiar non-relativistic equations of hydrostatic equilibrium. We now introduce the units (from Oppenheimer and Volkoff (1939)):

$$c = G = \frac{1}{8\pi}\frac{m_B c^2}{(\hbar/m_B c)^3} \tag{2}$$

(In these units the unit of mass is 9.29 $M_\odot$, the unit of length 13.69 km., and the unit of energy density is $(\frac{1}{6})$x(rest energy of a baryon)/(volume of a sphere of radius equal to Compton wavelength of baryon). We take

m_B as the neutron mass throughout.) Eqns.(1) become then the eqns. of Oppenheimer and Volkoff(1939):

$$\frac{du}{dr} = 4\pi\epsilon r^2$$

$$\frac{dt}{dr} = -4\pi r\left(\frac{dP}{dt}\right)^{-1}\frac{P+\epsilon}{1-2u/r}\left(P+u/(4\pi r^3)\right) \qquad (4)$$

with initial conditions

$$u = 0,\ t = t_o,\ \text{at } r = 0 \qquad (5)$$

corresponding to $M_r = 0$, $P = P_c$ in the nonrelativistic case. Solutions with $u < 0$ at the center cannot occur; see Oppenheimer and Volkoff (1939) r is the distance from the center, u a parameter which at the boundary takes on the value of the observable mass; and t is related to the (proper) particle density by

$$n = (3\pi^2)^{-1}(\hbar/m_B c)^{-3}\sinh^3(t/4) \qquad (6)$$

The representation of n was first used in stellar calculations by Chandrasekhar, m_B is the rest mass of one of the (identical) fermions** composing the star. ϵ and P are energy density and pressure, known functions of t*. We write

$$\epsilon = \epsilon_T + \epsilon_V\ ;\ P = P_T + P_V \qquad (7)$$

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* These known functions taken together constitute the equation of state.

** We take the mass of the (identical) fermions as being equal to the neutron mass m_B.

where ϵ_T is the kinetic energy density including rest energy density and ϵ_V the potential energy density, and similarly for P.

The units are those of Oppenheimer and Volkoff (1939).

III. Equations of State

In all cases

$$\epsilon_T = \frac{1}{4\pi}(\sinh t - t) \tag{8}$$

$$P_T = \frac{1}{12\pi}\left(\sinh t - 8\sinh\frac{t}{2} + 3t\right) \tag{9}$$

The expressions for ϵ_T and P_T are the same as that given by Landau and Lifshitz (1958) as one sees from Eqn. (2). It is not that given by Chandrasekhar (1957), since the rest energy density is included, as it must be; at nuclear densities omission of it would make the parametric form of the equation of state inaccurate:(cf. Saakyan 1964)*

a. Oppenheimer and Volkhoff (1939);

$$\epsilon_V = P_V = 0 \tag{10}$$

This is simply the case of neutrons without interactions.

b. Cameron (1959) as modified by Saakyan (1963);

*The non-relativistic limit, as one sees by expanding each expression in powers of t, is

$$P_T = \frac{2}{3}(\epsilon_T - m_B c^2 n)$$

$$\epsilon_V = \frac{1}{4\pi}\left(23.9 \sinh^8 \frac{t}{4} - 10.1 \sinh^6 \frac{t}{4}\right)$$

$$P_V = \frac{1}{4\pi}\left(39.9 \sinh^8 \frac{t}{4} - 10.1 \sinh^6 \frac{t}{4}\right) \tag{11}$$

This is derived from a nuclear potential given by Skyrme (1959) which is based on the many-body theory of nuclear matter. A three body effective potential is constructed from which the potential energy density is derived. The predictions of the potential agree well with the data from scattering experiments.

c. Zel'dovich (1959) discussed the hypothesis that the nuclear repulsion ("hard core") is due to a common fermion "inside" all baryons. This led to the conclusion that for different or identical baryons in S states there should appear a strong repulsion at small distnaces with a potential $\sim \hbar^2/(2\mu R^2)$ (μ the mass, R the separation).

We write for the two-body potential

$$V(r) = \frac{\hbar^2}{m_B} \eta(b-r) \frac{1}{r^2} \tag{12}$$

η is the step function and b is a number like the range. Assuming $n^{1/3} \gtrsim b^{-1}$ we get for the potential energy of one particle

$$E_{V,1} = 2\pi \frac{\hbar^2}{m_B} b n \tag{13}$$

We have not included the restriction to S states. In an attempt to do so, we say the average wave number, k_{AV}, satisfies

$$k_{AV}\, b = \alpha < 1 \tag{14}$$

We know that the Fermi wave number satisfies

$$\frac{k_{AV}}{k_F} = \beta < 1 \tag{15}$$

Then

$$b = \frac{\alpha}{\beta} k_F^{-1} \tag{16}$$

or

$$b = (3\pi^2)^{-1/3} \frac{\alpha}{\beta} n^{-1/3} < n^{-1/3} \tag{17}$$

which contradicts our previous assumption. We know, however, that this calculation will give the correct dependence of ϵ_V on n and may hope that the coefficient will not be too far off. Combining Eqns.(17) and (13) we get

$$E_{V,1} = (8\pi/3)^{1/3} \frac{\alpha}{\beta} \frac{\hbar^2}{m_B} n^{2/3} \tag{18}$$

for the energy of one particle.

Recalling $\epsilon_V = n E_{V,1}$, we obtain, in our units, just

$$\epsilon_V = \frac{16}{9\pi^2} \frac{\alpha}{\beta} \sinh^5 \frac{t}{4} \tag{19}$$

To get P_V, use

$$P_V = \sinh^6 \frac{t}{4} \frac{\partial}{\partial(\sinh^3 \frac{t}{4})} \left(\frac{\epsilon_V}{\sinh^3 \frac{t}{4}} \right) \tag{20}$$

to get

$$P_V = \frac{32}{27\pi^2} \frac{\alpha}{\beta} \sinh^5 \frac{t}{4} \tag{21}$$

d. Zel'dovich (1961) also considered the question of the most rigid equation of state compatible with the theory of relativity. He showed that baryons interacting via a vector field mediated by massive quanta resulted in (assuming $n^{-1/3} < \frac{1}{\mu}$)

$$E_{V,1} = 2\pi g^2 n^2/\mu^2 \tag{22}$$

$m = \hbar\mu$ is the mass of the meson and g is the baryonic charge of the baryons. (We take c = 1.) Again using $\epsilon_V = n E_{V,1}$ and Eqn.(20), and defining

$$\gamma = \frac{g^2/\hbar}{(m/m_B)^2} \tag{23}$$

we get (in our units) just

$$\epsilon_V = P_V = \frac{16}{9\pi^2} \gamma \sinh^6 \frac{t}{4} \tag{24}$$

Zel'dovich also assumed, and it is material to his argument, that

$$1 < \gamma < \left(\frac{m_B}{m}\right)^3 \tag{25}$$

He has taken, too, $\epsilon_T = m_B n$, but he states clearly that this is an approximation; for exactness we must use, and do use, Eqn.(8).

Note that for this equation of state the speed of sound is equal to that of light.

IV. Relativistic Restrictions on the Equation of State

The equation of state must always satisfy covariance requirements. In particular, for very large n, $P \propto n^s$, with $s > 2$, is not permitted; otherwise $v_{sound} > c$. Within this framework it is arguable that the equations given in c) and d) above are quite simple and have some justification from elementary particle theory. Their simplicity, and the radical differences in the dependence of P on n between them, make the unusually similar results for a), c), and d) (see Zel'dovich 1961) interesting.

V. Results

a. The Eqns.(4) were integrated numerically for 15 - 20 values of t_o, for each equation of state, on a 7094 computer. The initial conditions (Eqns.(5)) define the starting values, except in the case of infinite central density (see below); the integration is stopped when $t = 10^{-3}\ t_o$. The values of u and r at this point are M and R.

Note that we take $\frac{\alpha}{\beta} = 1$ and $\gamma = 3$ respectively, and that no values of t_o which would make $P(t_o) > \varepsilon(t_o)$ are used in the second equation of state.

The results are given in Fig. 1. and in the Table. The last value is also that for infinite density, except for the second equation of state, in which it is the last solution for which $P(t_o) < \varepsilon(t_o)$.

b. Infinite density solutions. It is possible, as Oppenheimer and Volkoff showed, to integrate the equations even if the central density is infinite. We obtain in the several cases the solutions:

$$\text{i)} \quad \frac{u}{r} = \frac{3}{14}, \quad \frac{1}{r^2} = \frac{7}{3} e^{t} \tag{26}$$

ii) ——

(See above; $P(t_o) > \epsilon(t_o)$ is not permitted)

$$\text{iii)} \quad \frac{u}{r} = \frac{12}{49}, \quad \frac{1}{r^2} = \frac{49}{54\pi} \frac{\alpha}{\beta} e^{5t/4} \tag{27}$$

$$\text{iv)} \quad \frac{u}{r} = \frac{1}{4}, \quad \frac{1}{r^2} = \frac{4}{9\pi} \gamma e^{3t/2} \tag{28}$$

These are taken in all cases as being accurate out to values of r such that t = 30, and the integration is done numerically from that point.

VI. Discussion

The general character of the M vs. t_o curves, Fig. 1, is the same for the first, third, and fourth equations of state. We have a gradual rise to an absolute maximum around $t_o = 2.5 - 3$, a decline to a relative minimum between 5.5 and 8.0, a rise to a relative maximum between 7.5 and 11, and a decline to the infinite density solution value, which is attained (to three significant figures) between 10 and 20.

The behavior of R as a function of t_o for each equation of state is quite similar to the curves 1a and 2a of Figure 2 of the paper by Arbartsumyan and Saakyan (1961).

We see that, except for the second equation of state (which for large t is unphysical), the M vs. t_o curves are quite similar. The features present in the solution curve for the simplest possible case (Oppenheimer and Volkoff) are present in the others also. In particular, there is an absolute maximum and a second relative maximum. One might think that this would indicate the presence of a second stability region from the first minimum to the second maximum: wherever the slope of the M versus t_o curve is positive the configuration should be stable. However, Misner and Zapolsky (to be published), following an earlier work by Chandrasekhar(1964 a) have shown that the equilibrium, for polytropes, is unstable in this range of values of the density. This result was also obtained by Chandrasekhar in a later addition (1964b).

Of course the existence of an absolute maximum for a given equation of state has the consequences pointed out by Oppenheimer and Volkoff (1939). No stable configuration is possible for $M > M_{max}$. A star having a greater mass must lose part of it by some mechanism or collapse towards the Schwarzchild singularity.

VII. Comparison with Previous Work

First equation of state: The curve of M vs. t_o does not agree with any published works (e.g. Oppenheimer and Volkoff 1939; Saakyan 1963), but Saakyan's (1962) comments seem to indicate that he is aware that two relative maxima must exist in this case also. Slight differences were found between the masses and radii of Oppenheimer and Volkoff (1939) and ours. Our value for the mass for $t_o = \infty$ does not agree with Oppenheimer and Volkoff (1939), but seems to be very close to the value given in Saakyan (1963). Note there cannot be any static approach to infinite density since maximum occurs before relative maximum. (See Zel'dovich 1959.)

Second equation of state: We confirm the results of Saakyan (1963) but numerical comparison could not be made, since Saakyan gave no table. Certain discrepancies exist between Cameron's (1959) results and ours, including one (for $\rho_c = 3\times10^{14}$ c.g.s.) in a region where Saakyan says his and Cameron's were identical. Note that Saakyan does not include one of Cameron's points in his graph, viz. $\rho_c = 2\times10^{14}$ c.g.s. ($t_o = 1.258$).

A Table is appended giving R, M, and M_N (see below) for each model and for each value of t_o.

VIII. Binding Energies

We have also calculated the quantity

$$u_N(R) = m_B c^2 \int_0^R 4\pi r^2 \left(1 - 2\frac{u}{r}\right)^{-1/2} n\, dr \tag{29}$$

where n is given by Eqn.(6), for all the models except the infinite density ones. The column M_N in the Table gives the values of $u_N(R)$ in solar mass units.

M_N is simply the total rest mass of the baryons composing the star. If $M_N - M$ is positive, we have binding, and conversely. The Table shows that, roughly speaking, for t_o somewhat past the first maximum, the configurations are unbound.

IX. Acknowledgement

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Figure Caption

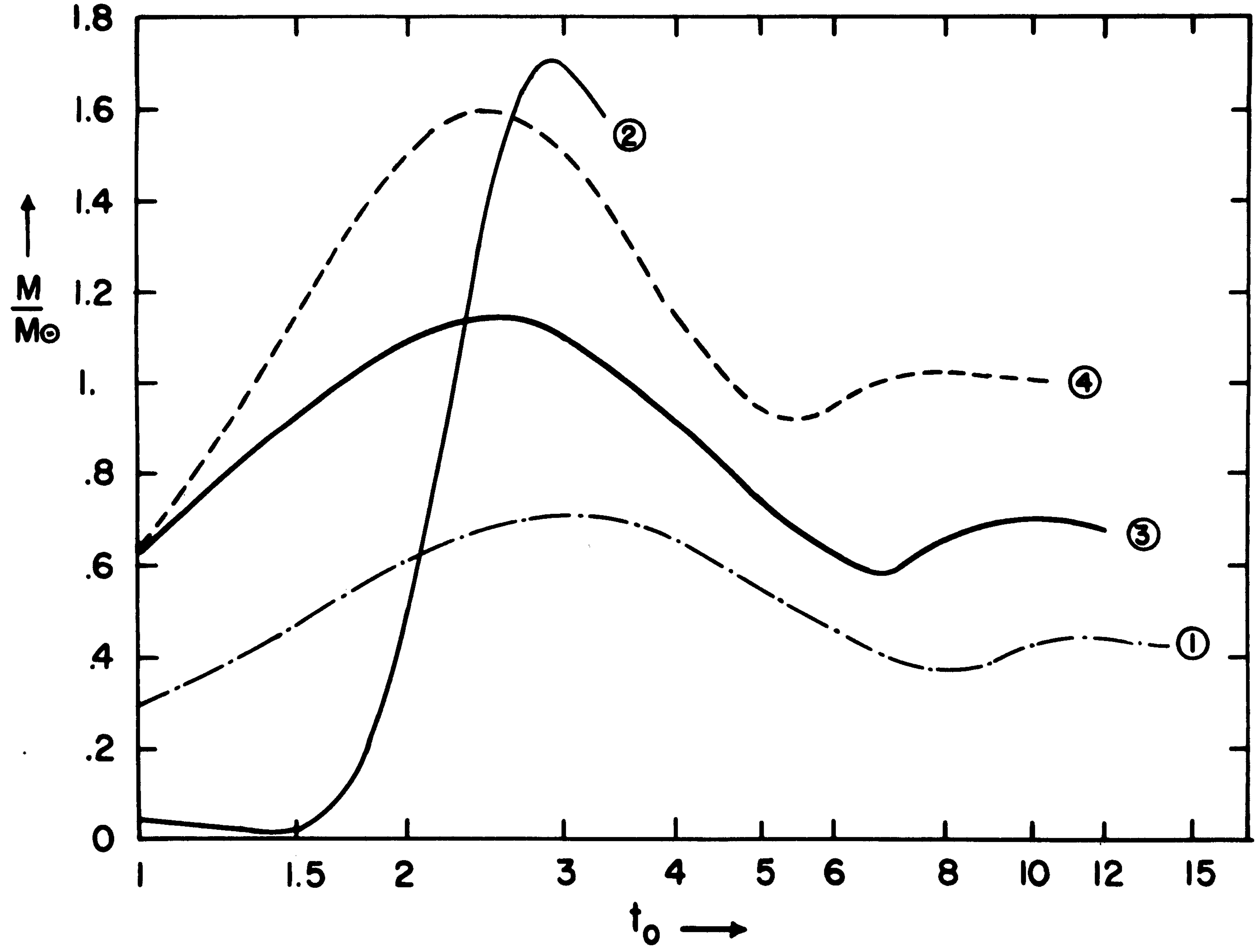
Figure 1.

1) First Equation of State, Eqn.(10)

2) Second Equation of State, Eqn.(11)

3) Third Equation of State, Eqns.(19-21) ($\frac{\alpha}{\beta} = 1$)

4) Fourth Equation of State, Eqn.(24) ($\gamma = 3$)



M/M⊙
1.8
1.6
1.4
1.2
1.
.8
.6
.4
.2
0
1
1.5
2
3
4
5
6
8
10
12
15
t0
2
4
3
1

Table 1.

First Equation of State, Eqn.(10)

t_o	M	M_N	R
1.0	.3015	.3043	20.76
1.5	.4793	.488	16.14
2.0	.6121	.6288	13.13
2.3	.6643	.6855	11.74
2.6	.6964	.7209	10.57
2.8	.7077	.7336	9.877
3.0	.7120	.7384	9.270
3.3	.7073	.7330	8.455
3.6	.6920	.7153	7.754
4.0	.6602	.6780	6.970
5.0	.5561	.5546	5.627
6.0	.4574	.4381	5.000
7.0	.3902	.3603	5.035
7.5	.3731	.3410	5.298
8.0	.3688	.3362	5.679
8.5	.3764	.3445	6.085
9.0	.3921	.3619	6.401
10.0	.4259	.3993	6.597
11.0	.4416	.4168	6.431
12.0	.4399	.4150	6.221
13.0	.4315	.4055	6.104
14.0	.4246	.3977	6.093
15.0	.4222	.3950	6.140
20.0	.4267	.4001	6.182
30.0	.4262	.3995	6.186
INFINITE	.4262		6.185

Second Equation of State, Eqn.(11)

t_o	M	M_N	R
0.5	.04571	.04575	27.14
1.0	.04489	.04493	22.09
1.3	.03707	.03709	22.41
1.5	.03339	.03342	8.518
1.6	.06479	.06513	6.904
1.8	.2093	.2140	7.147
2.0	.4955	.5220	8.010
2.2	.8931	.9797	8.633
2.4	1.286	1.473	8.821
2.5	1.442	1.685	8.765
2.6	1.560	1.854	8.624
2.7	1.640	1.974	8.428
2.8	1.685	2.044	8.194
2.9	1.701	2.070	7.940
3.0	1.694	2.058	7.681
3.2	1.636	1.958	7.184
3.3	1.594	1.886	6.960

Third Equation of State, Eqns.(19-21)
$(\frac{\alpha}{\beta} = 1)$

t_o	μ	M_N	R
1.0	.6292	.6385	26.53
1.5	.9325	.9582	20.17
2.0	1.100	1.142	16.03
2.2	1.131	1.176	14.73
2.4	1.143	1.190	13.60
2.5	1.144	1.191	13.08
2.6	1.141	1.188	12.60
2.8	1.126	1.171	11.71
3.0	1.103	1.143	10.93
4.0	.9165	.9170	8.237
5.0	.7297	.6905	7.064
6.0	.6124	5520	7.194
6.5	.5906	.5270	7.717
7.0	.5957	.5327	8.393
7.5	.6212	.5614	8.970
8.0	.6539	.5980	9.272
9.0	.6969	.6465	9.186
9.5	.7019	.6522	9.009
10.0	.6993	.6492	8.844
11.0	.6853	.6332	8.660
12.0	.6754	.6219	8.665
15.0	.6796	.6267	8.781
20.0	.6788	.6258	8.762
40.0	.6788	.6259	8.762
INFINITE	.6788		8.760

Fourth Equation of State, Eqn.(24)
$(\gamma = 3)$

t_o	M	M_N	R
1.0	.6232	.6341	24.18
1.3	.9459	.9749	21.02
1.5	1.148	1.194	19.25
1.7	1.320	1.387	17.64
2.0	1.503	1.598	15.50
2.2	1.569	1.677	14.23
2.3	1.587	1.699	13.64
2.4	1.595	1.709	13.08
2.5	1.595	1.709	12.55
2.7	1.573	1.681	11.58
3.0	1.501	1.586	10.35
3.2	1.436	1.501	9.659
3.5	1.331	1.359	8.824
3.7	1.259	1.264	8.390
4.0	1.159	1.130	7.914
4.5	1.024	.9531	7.568
5.0	.9440	.8512	7.733
5.3	.9252	.8279	8.017
5.5	.9242	.8266	8.242
5.7	.9307	.8345	8.467
6.0	.9502	.8582	8.755
7.0	1.018	.9405	9.037
8.0	1.027	.9521	8.845
9.0	1.015	.9368	8.747
10.0	1.010	.9305	8.765
12.0	1.013	.9342	8.794
30.0	1.013	.9338	8.788
INFINITE	1.013		8.787